On irrational or inverse Transformations of elliptic Functions *

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1.

We saw in the *Fundamenta nova*, whatever odd number n is, that it is possible to determine substitutions

$$y = \frac{x}{M} \cdot \frac{1 + A'x^2 + A''x^4 + \cdots + A^{\left(\frac{n-1}{2}\right)}x^{n-1}}{1 + B'x^2 + B''x^4 + \cdots + B^{\left(\frac{n-1}{2}\right)}x^{n-1}}$$

$$z = nMy \cdot \frac{1 + C'y^2 + C''y^4 + \cdots + C^{\left(\frac{n-1}{2}\right)}y^{n-1}}{1 + D'y^2 + D''y^4 + \cdots + D^{\left(\frac{n-1}{2}\right)}y^{n-1}}$$

of such a kind that

$$\frac{dy}{\sqrt{(1-yy)(1-\lambda^2 yy)}} = \frac{dx}{M\sqrt{(1-xx)(1-k^2xx)}}$$

$$\frac{dz}{\sqrt{(1-zz)(1-k^2)}} = \frac{nMdy}{\sqrt{(1-yy)(1-\lambda^2 yy)}}.$$

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We even gave general analytic expressions both for the applied substitutions and the transformed modulus λ. We will call these substitutions and transformations, which those expressions yield, rational or direct. In the following, we will teach, that not only general analytic expressions for these rational substitutions can be assigned, but also for irrational substitutions resulting from their inversion; for, one can generally and explicitly express x by y, y by z. I think this promotes algebraic analysis, since it could hardly solve this problem of such complexity and such huge generality and elegance before.
But before I consider this question, certain fundamental theorems are to be recalled, which we proved in earlier papers.
Having put

$$\int_0^u \Delta^2 \operatorname{am} u du = E(u) \qquad \int_0^u E(u) du = \log \Omega(u),$$

we saw in the *Commentatio prima* that it is possible to assign the constant r in infinitely many ways so that the function $e^{ruu}\Omega$, we called $\chi(u)$, becomes periodic, and it has the period it enjoys with the elliptic functions of the argument u in common. For, while m, m' denote positive or negative integers, we see, having put

$$mK + m'iK' = Q \qquad\qquad r = \frac{m'i\pi}{4KQ} - \frac{E}{2K'},$$

that

$$\chi(u + 4Q) = \chi(u).$$

But from the elements it is also known that

$$\sin \operatorname{am}(u+4Q) = \sin \operatorname{am} u, \quad \cos \operatorname{am}(u+4Q) = \cos \operatorname{am} u, \quad \Delta \operatorname{am}(u+4Q) = \Delta \operatorname{am} u \quad \text{etc.}$$

Vice verse, whatever period from the infinitely many ones, which are all composed of two, of the elliptic functions you chose, it is possible to determine the function $\chi(u)$, which enjoys the same period.
Further, in the said paper we demonstrated the fundamental formula:

$$(1.) \qquad \frac{\chi(u+a)\chi(u-a)}{\chi^2(a)\chi^2(u)} = 1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u,$$

and in the paper *Formula novae in theoria transcendentium ellipticarum fundamentales* we proved the formulas:

$$(2.)\qquad \frac{\chi(u+a)\chi(u+b)\chi(a+b)}{\chi(a)\chi(b)\chi(u)\chi(u+a+b)} = 1 + k^2 \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am} u \sin\operatorname{am}(u+a+b)$$

$$(3.)\qquad \sin\operatorname{am} a \sin\operatorname{am} b + \sin\operatorname{am} u \sin\operatorname{am}(u+a+b) - \sin\operatorname{am}(u+a) \sin\operatorname{am}(u+b)$$
$$= k^2 \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am} u \sin\operatorname{am}(u+a) \sin\operatorname{am}(u+b) \sin\operatorname{am}(u+a+b).$$

2.

Having mentioned these things in advance, while n denotes an arbitrary odd number, but m, m' arbitrary positive or negative numbers, which are nevertheless not divisible by the same factor of n, let us put

$$mK + m'iK' = Q = n\omega$$

and let us form the following expressions:

$$X = \sum \frac{\chi(u+4p\omega)}{\chi(u)\chi(4p\omega)} \cdot \sin\operatorname{am}(u+4p\omega)$$
$$Y = \sum \frac{\chi(u+4p\omega)}{\chi(u)\chi(4p\omega)} \cdot \frac{\cos\operatorname{am}(u+4p\omega)}{\Delta\operatorname{am} 4p\omega}$$
$$Z = \sum \frac{\chi(u+4p\omega)}{\chi(u)\chi(4p\omega)} \cdot \frac{\Delta\operatorname{am}(u+4p\omega)}{\cos\operatorname{am} 4p\omega},$$

in which sums one has to attribute the values 0, 1, 2, $\cdots$, $n-1$ to the number p. Therefore, having put $p = 0$ the first terms are:

$$\sin\operatorname{am} u, \quad \cos\operatorname{am} u, \quad \Delta\operatorname{am} u.$$

Let us multiply the expressions X, Y, Z by themselves first; further, let us form the product YZ.
Let us put

$$X_p = \frac{\chi(u+4p\omega)}{\chi(u)\chi(4p\omega)} \cdot \sin \operatorname{am}(u+4p\omega)$$
$$Y_p = \frac{\chi(u+4p\omega)}{\chi(u)\chi(4p\omega)} \cdot \frac{\cos \operatorname{am}(u+4p\omega)}{\Delta \operatorname{am} 4p\omega}$$
$$Z_p = \frac{\chi(u+4p\omega)}{\chi(u)\chi(4p\omega)} \cdot \frac{\Delta \operatorname{am}(u+4p\omega)}{\cos \operatorname{am} 4p\omega},$$

it will be

$$X = X_0 + X_1 + X_2 + \cdots + X_{n-1}$$
$$Y = Y_0 + Y_1 + Y_2 + \cdots + Y_{n-1}$$
$$Z = Z_0 + Z_1 + Z_2 + \cdots + Z_{n-1}.$$

The expressions X_p, Y_p, Z_p, since they consist of periodic functions, which remain unchanged having changed u into $u + 4Q$, are not changed, if p is changed into $p \pm n$. Hence it is possible to write X_{-h}, Y_{-h}, Z_{-h} instead of X_{n-h}, Y_{n-h}, Z_{n-h}. Having constituted these, let us put

$$(XX)_0 = X_0X_0 + 2X_1X_{-1} + 2X_2X_{-2} + \cdots + 2X_{\frac{n-1}{2}}X_{-\frac{n-1}{2}}$$
$$(YY)_0 = Y_0Y_0 + 2Y_1Y_{-1} + 2Y_2Y_{-2} + \cdots + 2Y_{\frac{n-1}{2}}Y_{-\frac{n-1}{2}}$$
$$(ZZ)_0 = Z_0Z_0 + ZX_1Z_{-1} + ZX_2Z_{-2} + \cdots + 2Z_{\frac{n-1}{2}}Z_{-\frac{n-1}{2}}$$

and in general

$$(XX)_p = X_0X_p + X_1X_{p-1} + X_2X_{p-2} + \cdots + X_{n-1}X_{p-n+1}$$
$$(YY)_p = Y_0Y_p + Y_1Y_{p-1} + Y_2Y_{p-2} + \cdots + Y_{n-1}Y_{p-n+1}$$
$$(ZZ)_p = Z_0Z_p + Z_1Z_{p-1} + Z_2Z_{p-2} + \cdots + Z_{n-1}Z_{p-n+1}$$

or:

$$(XX)_p = \sum X_hX_{p-h} \qquad (YY)_p = \sum Y_hY_{p-h} \qquad (ZZ)_p = \sum Z_hZ_{p-h},$$

if the values 0, 1, 2, $\cdots$, $n-1$ are attributed to h, it will be:

$$(4.)\qquad XX = (XX)_0 + (XX)_1 + (XX)_2 + \cdots (XX)_{n-1} = \sum (XX)_p$$
$$(5.)\qquad YY = (YY)_0 + (YY)_1 + (YY)_2 + \cdots (YY)_{n-1} = \sum (YY)_p$$
$$(6.)\qquad ZZ = (ZZ)_0 + (ZZ)_1 + (ZZ)_2 + \cdots (ZZ)_{n-1} = \sum (ZZ)_p.$$

3.

From the formulas, we gave in the *Fundamenta nova* (§ 18), it follows:

$$\sin\operatorname{am}(u+a)\sin\operatorname{am}(u-a) = \frac{\sin^2\operatorname{am}u - \sin^2\operatorname{am}a}{1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u}$$
$$\frac{\cos\operatorname{am}(u+a)\cos\operatorname{am}(u-a)}{\Delta^2\operatorname{am}a} = \frac{\cos^2\operatorname{am}u - \cos^2\operatorname{coam}a}{1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u}$$
$$\frac{\Delta\operatorname{am}(u+a)\Delta\operatorname{am}(u-a)}{\cos^2\operatorname{am}a} = \frac{\Delta^2\operatorname{am}u + k'k'\tan^2\operatorname{am}a}{1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u}$$

and hence from (1.):

$$X_h X_{-h} = \sin^2\operatorname{am}u - \sin^2\operatorname{am}4h\omega$$
$$Y_h Y_{-h} = \cos^2\operatorname{am}u - \cos^2\operatorname{coam}4h\omega$$
$$Z_h Z_{-h} = \Delta^2\operatorname{am}u + k'k'\tan^2\operatorname{am}4h\omega.$$

As in the *Commentatio prima* put:

$$\sin^2\operatorname{am}4\omega + \sin^2\operatorname{am}8\omega + \cdots + \sin^2\operatorname{am}2(n-1)\omega = \rho$$
$$\cos^2\operatorname{coam}4\omega + \cos^2\operatorname{coam}8\omega + \cdots + \cos^2\operatorname{coam}2(n-1)\omega = \sigma$$
$$k'k'[\tan^2\operatorname{am}4\omega + \tan^2\operatorname{am}8\omega + \cdots + \tan^2\operatorname{am}2(n-1)\omega] = \tau,$$

we find:

$$(7.)\qquad (XX)_0 = n\sin^2\operatorname{am}u - 2\rho$$
$$(8.)\qquad (YY)_0 = n\cos^2\operatorname{am}u - 2\sigma$$
$$(9.)\qquad (ZZ)_0 = n\Delta^2\operatorname{am}u + 2\tau.$$

4.

Before we investigate the values of the expressions $(XX)_p$, $(YY)_p$, $(ZZ)_p$ for the remaining values of p, let us transform the expressions Y_p, Z_p into a form similar to X_p. For this aim, let us expand the values of the expressions

$$\chi(u+K), \quad \chi(u+K+iK').$$

Let us by $G(u)$ denote the function

$$G(u) = \frac{\chi'(u)}{\chi(u)} = \frac{d\log\chi(u)}{du},$$

or, since

$$\chi(u) = e^{ruu}\Omega(u) \qquad \frac{d\log\Omega(u)}{du} = E(u),$$

the function

$$G(u) = 2ru + E(u).$$

Since $\chi(u+4Q) = \chi(u)$, it will also be

$$G(u+4Q) = G(u),$$

so that the function $G(u)$ also is periodic. Further, from the theorem on the addition of elliptic integrals of the second kind it follows:

$$(10.) \qquad G(u) + G(a) - G(u+a) = k^2 \sin\operatorname{am} a \sin\operatorname{am} u \sin\operatorname{am}(u+a),$$

whence, after having put $a = K$, $a = K + iK'$,

$$G(u+K) - G(K) - G(u) = -\frac{k^2 \sin\operatorname{am} u \cos\operatorname{am} u}{\Delta\operatorname{am} u} = \frac{d\log\Delta\operatorname{am} u}{du}$$

$$G(u+K+iK') - G(K+iK') - G(u) = -\frac{\sin\operatorname{am} u\Delta\operatorname{am} u}{\cos\operatorname{am} u} = \frac{d\log\cos\operatorname{am} u}{du},$$

from which formula, after an integration, it results:

$$\log \frac{\chi(u+K)}{\chi(u)\chi(K)} - G(K)\cdot u \qquad = \log \Delta \operatorname{am} u$$
$$\log \frac{\chi(u+K+iK'))}{\chi(u)\chi(K+iK')} - G(K+iK')\cdot u = \log\cos\operatorname{am} u$$

or

$$\frac{\chi(u+K)}{\chi(u)\chi(K)} = e^{G(K)\cdot u}\Delta \operatorname{am} u$$
$$\frac{\chi(u+K+iK')}{\chi(u)\chi(K+iK')} = e^{G(K+iK')\cdot u}\cos\operatorname{am} u$$

Hence, having put a and $u+a$ instead of u and after a division, it follows:

$$\frac{\chi(u+a+K)}{\chi(a+K)\chi(u)} = e^{G(K)\cdot u}\frac{\chi(u+a)}{\chi(a)\chi(u)}\cdot\frac{\Delta \operatorname{am}(u+a)}{\Delta \operatorname{am} a}$$
$$\frac{\chi(u+a+K+iK'))}{\chi(a+K+iK')\chi(u)} = e^{G(K+iK')\cdot u}\frac{\chi(u+a)}{\chi(a)\chi(u)}\cdot\frac{\cos\operatorname{am}(u+a)}{\cos\operatorname{am} a},$$

whence, having changed a into $a+K$, $a+K+iK'$, respectively, since

$$\Delta \operatorname{am} u \Delta \operatorname{am}(u+K) = k'$$
$$\cos\operatorname{am} u \cos\operatorname{am}(u+K+iK') = \frac{-ik'}{k}$$
$$\cos\operatorname{am} u \Delta \operatorname{am}(u+K+iK') = ik'\sin\operatorname{am} u,$$

which formulas are known from the elements (confer *Fund.* § 17, 19), one obtains:

$$\text{(11.)} \qquad \frac{\chi(u+a+2K))}{\chi(a+2K)\chi(u)} = e^{2G(K)\cdot u}\frac{\chi(u+a)}{\chi(a)\chi(u)}$$

$$\text{(12.)} \qquad \frac{\chi(u+a+2K+2iK')}{\chi(a+2K+2iK')\chi(u)} = e^{2G(K+iK')\cdot u}\frac{\chi(u+a)}{\chi(a)\chi(u)}$$

$$\text{(13.)} \qquad \frac{\chi(u+a+2K+iK')}{\chi(a+2K+iK')} = e^{[G(K)+G(K+iK')]u}\frac{\chi(u+a)}{\chi(a)\chi(u)}\cdot\frac{\sin\operatorname{am}(u+a)}{\sin\operatorname{am} a}.$$

Hence, since, having for the sake of brevity put $4p\omega = a$:

$$\begin{aligned} X_p &= \frac{\chi(u+a)}{\chi(a)\chi(u)}\cdot\sin\operatorname{am}(u+a) \\ Y_p &= \frac{\chi(u+a)}{\chi(a)\chi(u)}\cdot\frac{\cos\operatorname{am}(u+a)}{\Delta\operatorname{am} a} \\ Z_p &= \frac{\chi(u+a)}{\chi(a)\chi(u)}\cdot\frac{\Delta\operatorname{am}(u+a)}{\cos\operatorname{am} a}, \end{aligned}$$

having put

$$e^{-G(k)\cdot u} = \vartheta \qquad e^{-G(K+iK')\cdot u} = \vartheta'$$
$$a+k = a'. \qquad a+K+iK' = a''$$

we also have:

$$\begin{aligned} &\text{(14.)} \qquad X_p = \frac{\chi(u+a)}{\chi(a)\chi(u)}\cdot\sin\operatorname{am}(u+a) \\ &\text{(15.)} \qquad Y_p = \vartheta\cdot\frac{\chi(u+a')}{\chi(a')\chi(u)}\cdot\sin\operatorname{am}(u+a') \\ &\text{(16.)} \qquad Z_p = k\vartheta'\cdot\frac{\chi(u+a'')}{\chi(a'')\chi(u)}\cdot\sin\operatorname{am}(u+a''), \end{aligned}$$

whence, if we do not take the factors ϑ, $k\vartheta'$ common to all Y_p, Z_p into account, we obtain Y_p and Z_p from X_p by changing a to a' and a'', respectively.

5.

Having prepared all this, if one puts:

$$4h\omega = a \qquad 4(p-h)\omega = b$$
$$a + K = a' \qquad b + K = b'$$
$$a + K + iK' = a'' \qquad b + K + iK' = b'',$$

we find:

$$\begin{aligned}
X_h X_{p-h} &= \frac{\chi(u+a)}{\chi(a)\chi(u)} \cdot \frac{\chi(u+b)}{\chi(b)\chi(u)} \cdot \sin\mathrm{am}(u+a)\sin\mathrm{am}(u+b) \\
Y_h Y_{p-h} &= \vartheta\vartheta \frac{\chi(u+a')}{\chi(a)\chi(u)} \cdot \frac{\chi(u+b')}{\chi(b')\chi(u)} \cdot \sin\mathrm{am}(u+a')\sin\mathrm{am}(u+b') \\
Z_h Z_{p-h} &= k^2\vartheta'\vartheta' \frac{\chi(u+a'')}{\chi(a'')\chi(u)} \cdot \frac{\chi(u+b'')}{\chi(b'')\chi(u)} \cdot \sin\mathrm{am}(u+a'')\sin\mathrm{am}(u+b'').
\end{aligned}$$

Now one obtains from formula (2.):

$$\frac{\chi(u+a)}{\chi(a)\chi(u)} \cdot \frac{\chi(u+b)}{\chi(b)\chi(u)} = (1 + k^2 \sin\mathrm{am}\, a \sin\mathrm{am}\, b \sin\mathrm{am}\, u \sin\mathrm{am}(u+a+b)) \frac{\chi(u+a+b)}{\chi(a+b)\chi(u)};$$

further, from formula (3.):

$$\begin{aligned}
&\sin\mathrm{am}(u+a)\sin\mathrm{am}(u+b)(1 + k^2 \sin\mathrm{am} \sin\mathrm{am}\, b \sin\mathrm{am}\, u \sin\mathrm{am}(u+a+b)) \\
&= \sin\mathrm{am}\, a \sin\mathrm{am}\, b + \sin\mathrm{am}\, u \sin\mathrm{am}(u+a+b),
\end{aligned}$$

whence:

$$X_h X_{p-h} = \frac{\chi(u+a+b)}{\chi(a+b)\chi(u)} \cdot (\sin\mathrm{am}\, a \sin\mathrm{am}\, b + \sin\mathrm{am}\, u \sin\mathrm{am}(u+a+b)).$$

Hence, having changed a into a', a'', b into b', b'', also:

$$\begin{aligned}
Y_h Y_{p-h} &= \vartheta\vartheta \cdot \frac{\chi(u+a'+b')}{\chi(a'+b')\chi(u)} \cdot (\sin\mathrm{am}\, a' \sin\mathrm{am}\, b' + \sin\mathrm{am}\, u \sin\mathrm{am}(u+a'+b')) \\
Z_h Z_{p-h} &= k^2\vartheta'\vartheta' \cdot \frac{\chi(u+a''+b'')}{\chi(a''+b'')\chi(u)} \cdot (\sin\mathrm{am}\, a'' \sin\mathrm{am}\, b'' + \sin\mathrm{am}\, u \sin\mathrm{am}(u+a''+b'')).
\end{aligned}$$

But from (11.), (12.) :

$$\begin{aligned}
\vartheta\vartheta \cdot \frac{\chi(u+a'+b')}{\chi(a'+b')\chi(u)} &= \frac{\chi(u+a+b)}{\chi(a+b)\chi(u)} \\
\vartheta'\vartheta' \cdot \frac{\chi(u+a''+b'')}{\chi(a''+b'')\chi(u)} &= \frac{\chi(u+a+b)}{\chi(a+b)\chi(u)};
\end{aligned}$$

further,

$$\begin{aligned}
\sin\mathrm{am}(u+a'+b') &= -\sin\mathrm{am}(u+a+b) \\
\sin\mathrm{am}(u+a''+b'') &= -\sin\mathrm{am}(u+a+b),
\end{aligned}$$

whence, since $a+b=4p\omega$, having put h instead of 0, 1, 2, $\cdots$, $n-1$, after a summation it results:

$$\begin{aligned}
(XX)_p &= \sum X_h X_{p-h} = \frac{\chi(u+4p\omega)}{\chi(4p\omega)\chi(u)}[n \sin\mathrm{am}\, u \sin\mathrm{am}(u+4p\omega) + \sum \sin\mathrm{am}\, a \sin\mathrm{am}\, b] \\
(YY)_p &= \sum Y_h Y_{p-h} = \frac{\chi(u+4p\omega)}{\chi(4p\omega)\chi(u)}[-n \sin\mathrm{am}\, u \sin\mathrm{am}(u+4p\omega) + \sum \sin\mathrm{am}\, a' \sin\mathrm{am}\, b'] \\
(ZZ)_p &= \sum Z_h Z_{p-h} = k^2 \frac{\chi(u+4p\omega)}{\chi(4p\omega)\chi(u)}[-n \sin\mathrm{am}\, u \sin\mathrm{am}(u+4p\omega) + \sum \sin\mathrm{am}\, a'' \sin\mathrm{am}\, b''].
\end{aligned}$$

Therefore, the problem was reduced to the invention of the sums

$$\sum \sin\mathrm{am}\, a \sin\mathrm{am}\, b, \qquad \sum \sin\mathrm{am}\, a' \sin\mathrm{am}\, b', \qquad \sum \sin\mathrm{am}\, a'' \sin\mathrm{am}\, b''.$$

For this aim, let us note formula (10.):

$$\sin\mathrm{am}\, a \sin\mathrm{am}\, b = \frac{G(a)+G(b)-G(a+b)}{k^2 \sin\mathrm{am}(a+b)},$$

whence:

$$
\begin{aligned}
\sum \sin \operatorname{am} a \sin \operatorname{am} b &= \frac{\sum G(a) + \sum G(b) - nG(4p\omega)}{k^2 \sin 4p\omega} \\
\sum \sin \operatorname{am} a' \sin \operatorname{am} b' &= -\frac{\sum G(a') + \sum G(b') - nG(4p\omega + 2K)}{k^2 \sin \operatorname{am} 4p\omega} \\
\sum \sin \operatorname{am} a'' \sin \operatorname{am} b'' &= -\frac{\sum G(a'') + \sum G(b'') - nG(4p\omega + 2K + 2iK')}{k^2 \sin \operatorname{am} 4p\omega}.
\end{aligned}
$$

But

$$\sum G(a) = G(4\omega) + G(8\omega) + \cdots + G(4(n-1)\omega),$$

and since

$$G(4(n-1)\omega) = -G(4\omega), \qquad G(4(n-2)\omega) = -G(8\omega), \cdots$$

we have:

$$\sum G(a) = 0;$$

further, from (10.):

$$
\begin{aligned}
G(a') &= G(a) + G(K) - k^2 \sin \operatorname{am} a \sin \operatorname{coam} a \\
G(a'') &= G(a) + G(K + iK') - \frac{\sin \operatorname{am} a}{\sin \operatorname{coam} a},
\end{aligned}
$$

whence

$$
\begin{aligned}
\sum G(a') &= \sum G(a) + G(K) - k^2 \sum \sin \operatorname{am} a \sin \operatorname{coam} a \\
\sum G(a'') &= \sum G(a) + G(K + iK') - \sum \frac{\sin \operatorname{am} a}{\sin \operatorname{coam} a},
\end{aligned}
$$

and since the sums $\sum \sin \operatorname{am} a \sin \operatorname{coam} a$, $\sum \frac{\sin \operatorname{am} a}{\sin \operatorname{coam} a}$, while each two terms cancel, vanish, we find:

$$
\begin{aligned}
\sum G(a') &= nG(K) \\
\sum G(a'') &= nG(K + iK').
\end{aligned}
$$

The same way one finds:

$$\begin{aligned}\sum G(b) &= 0\\ \sum G(b') &= nG(k)\\ \sum G(b'') &= nG(K+iK').\end{aligned}$$

Furthermore, since:

$$\begin{aligned}G(4p\omega+2K) &= G(4p\omega)+2G(k)\\ G(4p\omega+2K+2iK') &= G(4p\omega)+2G(K+iK'),\end{aligned}$$

we have:

$$\begin{aligned}\sum \sin\operatorname{am} a \sin\operatorname{am} b &= -\frac{nG(4p\omega)}{k^2\sin\operatorname{am} 4p\omega}\\ \sum \sin\operatorname{am} a' \sin\operatorname{am} b'' &= \sum \sin\operatorname{am} a'' \sin\operatorname{am} b'' = \frac{nG(4p\omega)}{k^2\sin\operatorname{am} 4p\omega},\end{aligned}$$

whence finally

$$\begin{aligned}(XX)_P &= -(YY)_p = -\frac{1}{k^2}(ZZ)_p\\ &= n\left[\sin\operatorname{am} u \sin\operatorname{am}(u+4p\omega) - \frac{G(4p\omega)}{k^2\sin\operatorname{am} 4p\omega}\right]\frac{\chi(u+4p\omega)}{\chi(4p\omega)\chi(u)}.\end{aligned}$$

This formula, since:

$$G(4p\omega)+G(u)-G(u+4p\omega) = k^2\sin\operatorname{am} 4p\omega \sin\operatorname{am} u \sin\operatorname{am}(u+4p\omega),$$

can be exhibited more elegantly this way:

$$(17.)\qquad (XX)_p = -(YY)_P = -\frac{1}{k^2}(ZZ)_p = n\cdot\frac{G(u)-G(u+4p\omega)}{k^2\sin\operatorname{am} 4p\omega)}\cdot\frac{\chi(u+4p\omega)}{\chi(4p\omega)\chi(u)}.$$

But in this formula we assume that p is not $= 0$, from which case we found the formulas (7.), (8.), (9.).

6.

Having prepared these things, it follows from the formulas (4.), (5.), (6.):

$$(18.) \qquad XX + YY = n - 2\rho - 2\sigma$$

$$(19.) \qquad k^2XX + ZZ = n - 2k^2\rho + 2\tau.$$

For, from the formulas (7.), (8.), (9.):

$$(XX)_0 + (YY)_0 = n - 2\rho - 2\sigma$$

$$k^2(XX)_0 + (ZZ)_0 = n - 2k^2\rho + 2\tau;$$

further, from (17.):

$$(XX)_1 + (YY)_1 = 0, \qquad (XX)_2 + (YY)_2 = 0, \cdots$$

$$k^2(XX)_1 + (ZZ)_1 = 0, \qquad k^2(XX)_2 + (ZZ)_2 = \quad , \cdots$$

From formulas (18.), (19.), having put

$$X = \sqrt{n - 2\rho - 2\sigma} \cdot \sin\psi,$$

$$k^2 \cdot \frac{n - 2\rho - 2\sigma}{n - 2k^2\rho + 2\tau} = \lambda\lambda,$$

it follows that:

$$Y = \sqrt{n - 2\rho - 2\sigma} \cdot \cos\psi$$

$$Z = \sqrt{n - 2k^2\rho + 2\tau} \cdot \sqrt{1 - \lambda\lambda\sin^2\psi}.$$

Put

$$n - 2k^2\rho + 2\tau = \frac{1}{MM};$$

we saw that:

$$n - 2\rho - 2\sigma = \frac{\lambda^2}{k^2M^2},$$

whence:

$$X = \frac{\lambda}{kM}\sin\psi, \qquad Y = \frac{\lambda}{kM}\cos\psi, \qquad \frac{1}{M}\Delta(\psi, \lambda).$$

7.

Let us multiply the expressions

$$Y = Y_0 + Y_1 + Y_2 + \cdots + Y_{n-1}$$
$$Z = Z_0 + Z_1 + Z_2 + \cdots + Z_{n-1}$$

by each other. Let

$$(YZ)_p = \sum Y_h Z_{p-h}$$

while h denotes the numbers 0, 1, 2, $\cdots$ $n-1$; it will be

$$YZ = (YZ)_0 + (YZ)_1 + (YZ)_2 + \cdots + (YZ)_{n-1}.$$

Having again put

$$4h\omega = a \qquad 4(p-h)\omega = b$$
$$a + K = a' \qquad b + K + iK' = b'',$$

from formulas (15.), (16.) it follows:

$$Y_h Z_{p-h} = k\vartheta\vartheta' \cdot \frac{\chi(u+a')}{\chi(a')\chi(u)} \cdot \frac{\chi(u+b'')}{\chi(b'')\chi(u)} \cdot \sin\operatorname{am}(u+a')\sin\operatorname{am}(u+b'')$$

As above from (2.), (3.) we find this expression to be

$$= k\vartheta\vartheta' \cdot \frac{\chi(u+a'+b'')}{\chi(a'+b'')\chi(u)}[\sin\operatorname{am} a' \sin\operatorname{am} b'' + \sin\operatorname{am} u \sin(u+a'+b'')].$$

But from formula (13.):

$$\vartheta\vartheta' \cdot \frac{\chi(u+a'+b'')}{\chi(a'+b'')\chi(u)} = \frac{\chi(u+a+b)}{\chi(a+b)\chi(u)} \cdot \frac{\sin\operatorname{am}(u+a+b)}{\sin\operatorname{am}(a+b)};$$

further,

$$\sin\operatorname{am}(u+a'+b'') = \sin\operatorname{am}(u+a+b+2K+iK') = -\frac{1}{k\sin\operatorname{am}(u+a+b)},$$

whence, since $a + b = 4p\omega$, we find:

$$Y_h Z_{p-h} = \left[-\frac{\sin \operatorname{am} u}{\sin \operatorname{am} 4p\omega} + k\frac{k \sin \operatorname{am}(u + 4p\omega)}{\sin \operatorname{am} 4p\omega} \cdot \sin \operatorname{am} a' \sin \operatorname{am} b'' \right] \frac{\chi(u + 4p\omega)}{\chi(4p\omega)\chi(u)}$$

and hence, having put 0, 1, 2, $\cdots$, $n-1$ instead of h and having done the summation,

$$(YZ)_p = \sum Y_h Z_{p-h} = \left[-\frac{n \sin \operatorname{am} u}{\sin \operatorname{am} 4p\omega} + \frac{k \sin \operatorname{am}(u + 4p\omega)}{\sin \operatorname{am} 4p\omega} \sum \sin \operatorname{am} a' \sin \operatorname{am} b'' \right] \frac{\chi(u + 4p\omega)}{\chi(4p\omega)\chi(u)}$$

so that the task is reduced to finding the sum

$$\sum \sin a' \sin \operatorname{am} b''.$$

For this aim, I again note the formula

$$\sin \operatorname{am} a' \sin \operatorname{am} b'' = \frac{G(a') + G(b'') - G(a' + b'')}{k^2 \sin \operatorname{am}(a' + b'')},$$

whence, since:

$$\sin \operatorname{am}(a' + b'') = -\frac{1}{k \sin \operatorname{am} 4p\omega}$$

$$\sum G(a') = nG(K)$$

$$\sum G(b'') = nG(K + iK')$$

$$G(a' + b'') = G(4p\omega + 2K + iK') = G(4p\omega) + G(K) + G(K + iK') + \cot \operatorname{am} 4p\omega \Delta \operatorname{am} 4p\omega,$$

it results:

$$\sum \sin \operatorname{am} a' \sin \operatorname{am} b'' = \frac{n \sin \operatorname{am} 4p\omega}{k} [\cot \operatorname{am} 4p\omega \Delta \operatorname{am} 4p\omega + G(4p\omega)].$$

Having collected all these formulas we finally obtain:

$$(20.) \qquad (YZ)_p = n \left[\frac{\cos \operatorname{am} 4p\omega \Delta \operatorname{am} 4p\omega \sin \operatorname{am}(u + 4p\omega) - \sin \operatorname{am} u}{\sin \operatorname{am} 4p\omega} + \sin \operatorname{am}(u + 4p\omega) G(4p\omega) \right] \frac{\chi(u + 4p\omega)}{\chi(4p\omega)\chi(u)}.$$

We assume p not to be $= 0$ in this formula, which cases need some special consideration.
To find the value of

$$(YZ)_0 = Y_0Z_0 + Y_1Z_{-1} + Y_{-1}Z_1 + Y_2Z_{-2} + Y_{-2}Z_2 + \cdots + Y_{\frac{n-1}{2}}Z_{-\frac{n-1}{2}} + Y_{-\frac{n-1}{2}}Z_{\frac{n-1}{2}},$$

having recalled the formula

$$\frac{\cos\operatorname{am}(u+a)\Delta\operatorname{am}(u-a)}{\cos\operatorname{am}a\Delta\operatorname{am}a} = \frac{\cos\operatorname{am}u\Delta\operatorname{am}u - k'k'\frac{\tan\operatorname{am}a}{\Delta\operatorname{am}a}\cdot\sin\operatorname{am}u}{1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u},$$

we conclude from (1.):

$$Y_hZ_{-h} = \cos\operatorname{am}u\Delta\operatorname{am}u \;-\; k'k'\frac{\tan\operatorname{am}4h\omega}{\Delta\operatorname{am}4h\omega}\cdot\sin\operatorname{am}u,$$

$$Y_{-h}Z_h = \cos\operatorname{am}u\Delta\operatorname{am}u \;+\; k'k'\frac{\tan\operatorname{am}4h\omega}{\Delta\operatorname{am}4h\omega}\cdot\sin\operatorname{am}u,$$

whence:

$$(YZ)_0 = n\cos\operatorname{am}u\Delta\operatorname{am}u.$$

Expression (20.) can be transformed even further by means of the formulas

$$\begin{aligned}\sin\operatorname{am} &= \sin\operatorname{am}(u+4p\omega-4p\omega)\\ &= \frac{\sin\operatorname{am}(u+4p\omega)\cos\operatorname{am}4p\omega\Delta\operatorname{am}4p\omega - \sin\operatorname{am}4p\omega\cos\operatorname{am}(u+4p\omega)\Delta(u+4p\omega)}{1-k^2\sin^2\operatorname{am}4p\omega\sin^2\operatorname{am}(u+4p\omega)}\end{aligned}$$

$$k^2\sin\operatorname{am}4p\omega\sin\operatorname{am}u\sin\operatorname{am}(u+4p\omega) = G(4p\omega)+G(u)-G(u+4p\omega),$$

having added which:

$$\frac{\cos\operatorname{am}4p\omega\Delta\operatorname{am}4p\omega\sin\operatorname{am}(u+4p\omega) - \sin\operatorname{am}u}{\sin\operatorname{am}4p\omega}$$

$$= \cos\operatorname{am}(u+4p\omega)\Delta\operatorname{am}(u+4p\omega) - k^2\sin\operatorname{am}4p\omega\sin\operatorname{am}u\sin^2\operatorname{am}(u+4p\omega)$$

$$= \cos\operatorname{am}(u+4p\omega)\Delta\operatorname{am}(u+4p\omega) + [G(u+4p\omega)-G(u)-G(4p\omega)]\sin\operatorname{am}(u+4p\omega),$$

whence

$$(YZ)_p = n[\cos\operatorname{am}(u+4p\omega)\Delta\operatorname{am}(u+4p\omega)+\sin\operatorname{am}(u+4p\omega)(G(u+4p\omega)-G(u))]\frac{\chi(u+4p\omega)}{\chi(4p\omega)\chi(u)},$$

which formula also holds for $p = 0$.
We already noted that:

$$\begin{aligned}\frac{d\frac{\chi(u+4p\omega)}{\chi(4p\omega)\chi(u)}}{du} &= \frac{\chi(u+4p\omega)}{\chi(4p\omega)\chi(u)}\left[\frac{d\log\chi(u+4p\omega)}{du}-\frac{d\log\chi(u)}{du}\right]\\ &= \frac{\chi(u+4p\omega)}{\chi(4p\omega)\chi(u)}[G(u+4p\omega)-G(u)],\end{aligned}$$

whence, since further

$$\frac{d\sin\operatorname{am}(u+4p\omega)}{du} = \cos\operatorname{am}(u+4p\omega)\Delta\operatorname{am}(u+4p\omega),$$

we find:

$$(YZ)_p = n\frac{d\sin\operatorname{am}(u+4p\omega)\frac{\chi(u+4p\omega)}{\chi(4p\omega)\chi(u)}}{du} = n\frac{dX_p}{du}.$$

Hence

$$\sum(YZ)_p = n\sum\frac{dX_P}{du}$$

or

$$(21.)\qquad YZ = n\frac{dX}{du}.$$

8.

But now we found, having put

$$X = \frac{\lambda}{kM}\sin\psi,$$

that

$$Y = \frac{\lambda}{kM}\cos\psi,\qquad Z = \frac{1}{M}\Delta(\psi,\lambda),$$

whence equation (21.) goes over into:

$$\frac{\cos\psi\Delta(\psi,\lambda)}{M} = n\frac{d\sin\psi}{du}$$

or

$$\frac{d\psi}{du} = \frac{\Delta(\psi,\lambda)}{nM}, \qquad \frac{du}{nM} = \frac{d\psi}{\sqrt{1-\lambda\lambda\sin^2\psi}}.$$

Hence, since ψ and u vanish at the same time, in Legendre's notation it will be

$$\frac{u}{nM} = F(\psi,\lambda)$$

or in our notation

$$\psi = \mathrm{am}\left(\frac{u}{nM},\lambda\right),$$

whence

$$X = \frac{\lambda}{kM}\sin\mathrm{am}\left(\frac{u}{nM},\lambda\right)$$

$$Y = \frac{\lambda}{kM}\cos\mathrm{am}\left(\frac{u}{nM},\lambda\right)$$

$$Z = \frac{1}{M}\Delta\,\mathrm{am}\left(\frac{u}{nM},\lambda\right).$$

Hence follow the

Fundamental Formulas

(22.) $$\frac{\lambda}{kM}\sin\mathrm{am}\left(\frac{u}{nM},\lambda\right) = \sin\mathrm{am}\,u + \sum\sin\mathrm{am}(u+4\nu\omega)\,\frac{\chi(u+4\nu\omega)}{\chi(4\nu\omega)\chi(u)}$$

(23.) $$\frac{\lambda}{kM}\cos\mathrm{am}\left(\frac{u}{nM},\lambda\right) = \cos\mathrm{am}\,u + \sum\frac{\sin\mathrm{am}(u+4\nu\omega)}{\Delta\,\mathrm{am}\,4\nu\omega}\,\frac{\chi(u+4\nu\omega)}{\chi(4\nu\omega)\chi(u)}$$

(24.) $$\frac{1}{M}\Delta\,\mathrm{am}\left(\frac{u}{nM},\lambda\right) = \Delta\,\mathrm{am}\,u + \sum\frac{\Delta\,\mathrm{am}(u+4\nu\omega)}{\cos\mathrm{am}\,4\nu\omega}\,\frac{\chi(u+4\nu\omega)}{\chi(4\nu\omega)\chi(u)},$$

if one attributes the values $\pm1, \pm2, \cdots, \pm\frac{n-1}{2}$ to the number ν. One has to add the following formula, which flows from (7.), (17.), to these:

$$(25.)\quad \frac{\lambda^2}{k^2M^2}\sin^2\operatorname{am}\left(\frac{u}{nM},\lambda\right) = n\sin^2\operatorname{am}u - 2\rho + n\sum\frac{G(u)-G(u+4\nu\omega)}{k^2\sin\operatorname{am}4\nu\omega}\cdot\frac{\chi(u+4\nu\omega)}{\chi(4\nu\omega)\chi(u)}$$

$$= n\sin^2\operatorname{am}u - 2\rho + n\sum\left[\sin\operatorname{am}u\sin\operatorname{am}(u+4\nu\omega) - \frac{G(4\nu\omega)}{k^2\sin\operatorname{am}4\nu\omega}\right]\frac{\chi(u+4\nu\omega)}{\chi(4\nu\omega)\chi(u)}.$$

It is convenient to note that the modulus λ and the multiplicator M correspond to a transformation of n-th order corresponding to the variable ω. For, from (23.), (24.), if one puts $u = 0$:

$$\frac{\lambda}{kM} = 1 + \sin\operatorname{coam}4\omega + \sin\operatorname{coam}8\omega + \cdots + \sin\operatorname{coam}4(n-1)\omega$$
$$\frac{1}{M} = 1 + \frac{1}{\sin\operatorname{coam}4\omega} + \frac{1}{\sin\operatorname{coam}8\omega} + \cdots + \frac{1}{\sin\operatorname{coam}4(n-1)\omega};$$

but the same formulas result from the formula (*Fund.* § 23 (16.)):

$$\frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) = \sin\operatorname{am}u + \sin\operatorname{am}(u+4\omega) + \cdots + \sin\operatorname{am}(u+4(n-1)\omega),$$

if one puts $u = K$, $u = K + iK'$, respectively.
Formulas (22.) – (25.), since

$$\chi(u+4Q) = \chi(u+4n\omega) = \chi(u)$$
$$G(u+4Q) = G(u+4n\omega) = G(u),$$

remain unchanged, having changed u into $u + 4Q$ or into $u + 4pQ$, while p denotes an arbitrary positive or negative number. But if we assume some numbers m, m' and they do not have a common factor, what is possible without loss of generality, one is able to determine positive or negative numbers μ, μ' of such a kind that

$$m\mu' - \mu m' = 1;$$

having done this, if one puts

$$\mu K + \mu' iK' = Q' = 4n\omega',$$

$4Q'$ will be the period conjugated to $4Q$ and according to equation (32.) of the *Commentatio prima*

$$\frac{\chi(u+4p\omega+4p'Q')}{\chi(u+4p'Q')} = e^{-\frac{8pp'i\pi}{n}}\frac{\chi(u+4p\omega)}{\chi(u)}.$$

Hence, having changed u into $u+4pQ'$, from formulas (22.) − (25.):

$$(26.)\qquad \frac{\lambda}{kM}\ \sin\operatorname{am}\left(\frac{u+4pQ'}{nM},\lambda\right) = \sin\operatorname{am} u + \sum e^{-\frac{8r p\pi i}{n}}\sin\operatorname{am}(u+4\nu\omega)\cdot\frac{\chi(u+4\nu\omega)}{\chi(4\nu\omega)\chi(u)}$$

$$(27.)\qquad \frac{\lambda}{kM}\ \sin\operatorname{am}\left(\frac{u+4pQ'}{nM},\lambda\right) = \cos\operatorname{am} u + \sum e^{-\frac{8r p\pi i}{n}}\frac{\cos\operatorname{am}(u+4\nu\omega)}{\Delta\operatorname{am}4\nu\omega}\cdot\frac{\chi(u+4\nu\omega)}{\chi(4\nu\omega)\chi(u)}$$

$$(28.)\qquad \frac{1}{M}\quad \Delta\operatorname{am}\left(\frac{u+4pQ'}{nM},\lambda\right) = \Delta\operatorname{am} u + \sum e^{-\frac{8r p\pi i}{n}}\frac{\Delta\operatorname{am}(u+4\nu\omega)}{\cos\operatorname{am}4\nu\omega}\cdot\frac{\chi(u+4\nu\omega)}{\chi(4\nu\omega)\chi(u)}$$

$$(29.)\qquad \frac{\lambda^2}{k^2M^2}\sin^2\operatorname{am}\left(\frac{u+4pQ'}{nM},\lambda\right) = n\sin^2\operatorname{am} u - 2\rho + n\sum e^{-\frac{8\nu p\pi i}{n}}\frac{G(u)-G(u+4\nu\omega)}{k^2\sin\operatorname{am}4\nu\omega}\cdot\frac{\chi(u+4\nu\omega)}{\chi(4\nu\omega)\chi(u)}.$$

After we substitute the values 0, 1, 2, $\cdots$, $n-1$ for p in these formulas, we obtain four systems of equations, from which one easily finds the formulas:

$$(30.)\qquad \sin\operatorname{am} u = \frac{\lambda}{nkM}\ \sum\ \sin\operatorname{am}\left(\frac{u+4pQ'}{nM},\lambda\right)$$

$$(31.)\qquad \cos\operatorname{am} u = \frac{\lambda}{nkM}\ \sum\ \cos\operatorname{am}\left(\frac{u+4pQ'}{nM},\lambda\right)$$

$$(32.)\qquad \Delta\operatorname{am} u = \frac{1}{nM}\ \sum\ \Delta\operatorname{am}\left(\frac{u+4pQ'}{nM},\lambda\right)$$

$$(33.)\qquad \sin^2\operatorname{am} u - \frac{2\rho}{n} = \frac{\lambda^2}{n^2k^2M^2}\sum\sin^2\operatorname{am}\left(\frac{u+4pQ'}{nM},\lambda\right)$$

or these more general ones:

$$\text{(34.)} \qquad \sin\operatorname{am}(u+4\nu\omega)\cdot\frac{\chi(u+4\nu\omega)}{\chi(4\nu\omega)\chi(u)} = \frac{\lambda}{nkM}\sum e^{\frac{8\nu p\pi i}{n}}\sin\operatorname{am}\left(\frac{u+4pQ'}{nM},\lambda\right)$$

$$\text{(35.)} \qquad \frac{\cos\operatorname{am}(u+4\nu\omega)}{\Delta\operatorname{am}4\nu\omega}\cdot\frac{\chi(u+4\nu\omega)}{\chi(4\nu\omega)\chi(u)} = \frac{\lambda}{nkM}\sum e^{\frac{8\nu p\pi i}{n}}\cos\operatorname{am}\left(\frac{u+4pQ'}{nM},\lambda\right)$$

$$\text{(36.)} \qquad \frac{\Delta\operatorname{am}(u+4\nu\omega)}{\cos\operatorname{am}4\nu\omega}\cdot\frac{\chi(u+4\nu\omega)}{\chi(4\nu\omega)\chi(u)} = \frac{1}{nM}\sum e^{\frac{8\nu p\pi i}{n}}\Delta\operatorname{am}\left(\frac{u+4pQ'}{nM},\lambda\right)$$

$$\text{(37.)} \qquad \frac{G(u)-G(u+4\nu\omega)}{k^2\sin\operatorname{am}4\nu\omega}\cdot\frac{\chi(u+4\nu\omega)}{\chi(4\nu\omega)\chi(u)} = \frac{\lambda^2}{n^2k^2M^2}\sum e^{\frac{8\nu p\pi i}{n}}\sin^2\operatorname{am}\left(\frac{u+4pQ'}{nM},\lambda\right)$$

9.

Having put

$$r' = \frac{\mu'\pi i}{4KQ'} - \frac{E}{2K}$$

$$\chi\left(\frac{u}{M},\lambda\right) = e^{(m'-\tau)uu}\Omega\left(\frac{u}{M},\lambda\right)$$

$$G\left(\frac{u}{M},\lambda\right) = \frac{\chi'\left(\frac{u}{M},\lambda\right)}{\chi\left(\frac{u}{M},\lambda\right)},$$

we find:

$$\chi\left(\frac{u+4Q'}{M},\lambda\right) = \chi\left(\frac{u}{M},\lambda\right), \qquad G\left(\frac{u+4Q'}{M},\lambda\right) = G\left(\frac{u}{M},\lambda\right)$$

and from the formulas (22.) − (25.), having changed k into λ, λ into k, M into $\frac{(-1)^{\frac{n-1}{2}}}{nM}$, u into $\frac{u}{M}$, ω into $\frac{\omega'}{M}$, ρ into

$$\rho' = \sin^2\operatorname{am}\left(\frac{2\omega'}{M},\lambda\right) + \sin^2\operatorname{am}\left(\frac{4\omega'}{M},\lambda\right) + \cdots + \sin^2\operatorname{am}\left(\frac{(n-1)\omega'}{M},\lambda\right),$$

one obtains the following:

$$(38.)\quad \frac{nkM}{\lambda}\sin\operatorname{am}u = \sin\operatorname{am}\left(\frac{u}{M},\lambda\right) + \sum \sin\operatorname{am}\left(\frac{u+4\nu\omega'}{M},\lambda\right)\cdot\frac{\chi\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\chi\left(\frac{4\nu\omega'}{M},\lambda\right)\chi\left(\frac{u}{M},\lambda\right)}$$

$$(39.)\quad \frac{(-1)^{\frac{n-1}{2}}nkM}{\lambda}\cos\operatorname{am}u = \cos\operatorname{am}\left(\frac{u}{M},\lambda\right) + \sum \frac{\cos\operatorname{am}\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\Delta\operatorname{am}\left(\frac{4\nu\omega'}{M},\lambda\right)}\cdot\frac{\chi\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\chi\left(\frac{4\nu\omega'}{M},\lambda\right)\chi\left(\frac{u}{M},\lambda\right)}$$

$$(40.)\quad (-1)^{\frac{n-1}{2}}nM\Delta\operatorname{am}u = \Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) + \sum \frac{\Delta\operatorname{am}\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\cos\operatorname{am}\left(\frac{4\nu\omega'}{M},\lambda\right)}\cdot\frac{\chi\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\chi\left(\frac{4\nu\omega'}{M},\lambda\right)\chi\left(\frac{u}{M},\lambda\right)}$$

$$(41.)\quad \frac{n^2k^2M^2}{\lambda^2}\sin^2\operatorname{am}u = n\sin^2\operatorname{am}\left(\frac{u}{M},\lambda\right) - 2\rho' + \sum \frac{\left(\frac{u}{M},\lambda\right) - G\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\lambda^2\sin\operatorname{am}\left(\frac{4\nu\omega'}{M},\lambda\right)}\cdot\cdot\frac{\chi\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\chi\left(\frac{4\nu\omega'}{M},\lambda\right)\chi\left(\frac{u}{M},\lambda\right)}$$

From these, since

$$\frac{\chi\left(\frac{u+4\nu\omega'+4p\omega}{M},\lambda\right)}{\chi\left(\frac{u+4p\omega}{M},\lambda\right)} = e^{\frac{8\nu p\pi i}{n}}\frac{\chi\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\chi\left(\frac{u}{M},\lambda\right)},$$

having changed u into $u+4p\omega$, these more general formulas follow:

$$(42.)\quad \frac{nkM}{\lambda}\sin\operatorname{am}(u+4p\omega) = \sin\operatorname{am}\left(\frac{u}{M},\lambda\right) + \sum e^{\frac{8\nu p\pi i}{n}}\sin\operatorname{am}\left(\frac{u+4\nu\omega'}{M},\lambda\right)\cdot\frac{\chi\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\chi\left(\frac{4\nu\omega'}{M},\lambda\right)\chi\left(\frac{u}{M},\lambda\right)}$$

$$(43.)\quad \frac{(-1)^{\frac{n-1}{2}}nkM}{\lambda}\cos\operatorname{am}(u+4p\omega) = \cos\operatorname{am}\left(\frac{u}{M},\lambda\right) + \sum e^{\frac{8\nu p\pi i}{n}}\frac{\cos\operatorname{am}\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\Delta\operatorname{am}\left(\frac{4\nu\omega'}{M},\lambda\right)}\cdot\frac{\chi\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\chi\left(\frac{4\nu\omega'}{M},\lambda\right)\chi\left(\frac{u}{M},\lambda\right)}$$

$$(44.)\quad (-1)^{\frac{n-1}{2}}nM\Delta\operatorname{am}(u+4p\omega) = \Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) + \sum e^{\frac{8\nu p\pi i}{n}}\frac{\Delta\operatorname{am}\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\cos\operatorname{am}\left(\frac{4\nu\omega'}{M},\lambda\right)}\cdot\frac{\chi\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\chi\left(\frac{4\nu\omega'}{M},\lambda\right)\chi\left(\frac{u}{M},\lambda\right)}$$

$$(45.)\quad \frac{n^2k^2M^2}{\lambda^2}\sin^2\operatorname{am}(u+4p\omega) = n\sin^2\operatorname{am}\left(\frac{u}{M},\lambda\right) - 2\rho' + \sum e^{\frac{8\nu p\pi i}{n}}\frac{\left(\frac{u}{M},\lambda\right) - G\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\lambda^2\sin\operatorname{am}\left(\frac{4\nu\omega'}{M},\lambda\right)}\cdot\cdot\frac{\chi\left(\frac{u+4\nu\omega'}{M},\lambda\right)}{\chi\left(\frac{4\nu\omega'}{M},\lambda\right)\chi\left(\frac{u}{M},\lambda\right)}$$

10.

Having put

$$x = \sin\operatorname{am} u \quad \text{and}$$

$$(46.)\quad y = \sin\left(\frac{u}{M}, \lambda\right) = \frac{\frac{x}{M}\left(1 - \frac{xx}{\sin^2\operatorname{am}2\omega}\right)\left(1 - \frac{xx}{\sin^2\operatorname{am}4\omega}\right)\cdots\left(1 - \frac{xx}{\sin^2\operatorname{am}(n-1)\omega}\right)}{(1 - k^2\sin^2\operatorname{am}2\omega\cdot xx)(1 - k^2\sin^2\operatorname{am}4\omega\cdot xx)\cdots(1 - k^2\sin^2\operatorname{am}(n-1)\omega\cdot xx)}$$

$$(47.)\quad z = \sin\operatorname{am} nu = \frac{nMy\left(1 - \frac{yy}{\sin^2\operatorname{am}\left(\frac{2\omega'}{M},\lambda\right)}\right)\left(1 - \frac{yy}{\sin^2\operatorname{am}\left(\frac{4\omega'}{M},\lambda\right)}\right)\cdots\left(1 - \frac{yy}{\sin^2\operatorname{am}\left(\frac{(n-1)\omega'}{M},\lambda\right)}\right)}{\left(1 - \lambda^2\sin^2\operatorname{am}\left(\frac{2\omega'}{M},\lambda\right)yy\right)\left(1 - \lambda^2\sin^2\operatorname{am}\left(\frac{4\omega'}{M},\lambda\right)yy\right)\cdots\left(1 - \lambda^2\sin^2\operatorname{am}\left(\frac{(n-1)\omega'}{M},\lambda\right)yy\right)}$$

$$\Phi(u) = (1 - k^2\sin^2\operatorname{am}2\omega\cdot xx)(1 - k^2\sin^2\operatorname{am}4\omega\cdot xx)\cdots(1 - k^2\sin^2\operatorname{am}(n-1)\omega\cdot xx)$$

$$\Psi(u) = \left(1 - \lambda^2\sin^2\operatorname{am}\left(\frac{2\omega'}{M},\lambda\right)yy\right)\left(1 - \lambda^2\sin^2\operatorname{am}\left(\frac{4\omega'}{M},\lambda\right)yy\right)\cdots\left(1 - \lambda^2\sin^2\operatorname{am}\left(\frac{(n-1)\omega'}{M},\lambda\right)yy\right)$$

according to formula (17.) of *Commentatio prima*:

$$\frac{\chi(u + 4p\omega)}{\chi(4p\omega)\chi(u)} = \sqrt[n]{\frac{\Phi(4p\omega)\Phi(u)}{\Phi(u + 4p\omega)}}, \qquad \frac{\chi\left(\frac{u+4p\omega'}{M},\lambda\right)}{\chi\left(\frac{4p\omega'}{M},\lambda\right)\chi\left(\frac{u}{M},\lambda\right)} = \sqrt[n]{\frac{\Psi(4p\omega')\Psi(u)}{\Psi(u + 4p\omega')}}.$$

If these expressions are substituted in the equations (26.), (42.) and additionally one puts nu instead of u in (26.), it results:

$$(48.)\qquad \frac{nkM}{\lambda}\sin\operatorname{am}(u + 4p\omega) = \sin\operatorname{am}\left(\frac{u}{M},\lambda\right) + \sum e^{\frac{8\nu p\pi i}{n}}\sin\operatorname{am}\left(\frac{u + 4\nu\omega'}{M},\lambda\right)\cdot\sqrt[n]{\frac{\Psi(4\nu\omega')\Psi(u)}{\Psi(u + 4\nu\omega')}}$$

$$(49.)\qquad \frac{\lambda}{kM}\sin\left(\frac{u + 4p\omega'}{M},\lambda\right) = \sin\operatorname{am} u + \sum e^{\frac{-8\nu p\pi i}{n}}\sin\operatorname{am}(nu + 4\nu\omega)\cdot\sqrt[n]{\frac{\Phi(4\nu\omega)\Phi(nu)}{\Phi(nu + 4\nu\omega)}}.$$

The first of these equations yields a general explicit expression of x in terms of y or the complete algebraic resolution of the equation of n-th order (46.), but the other gives a general explicit expression of y in terms of z or the algebraic resolution of equation (47.). One has to note that by means of equation (2.) all radicals in one of these equations can be expressed by the power of one of them.